A FASTER PSEUDOPOLYNOMIAL TIME ALGORITHM FOR SUBSET SUM

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SODA 2017

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INTRODUCTION

Input: A set S of n positive integers $x_1, x_2, x_3, \ldots, x_n$ and a positive target integer t.

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Output: Is there a subset T of S such that $\sum_{x_i \in T} x_i = t$?

Classical problem.

THE SUBSET SUM PROBLEM

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Textbook DP algorithm due to Bellman that runs in $O(nt)$ **pseudopolynomial** time.

[Bellman '56]

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APPLICATIONS

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- set-based queries in databases
- applications in security

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PREVIOUS WORK

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- First poly space algorithm: $\widetilde{O}(n^3 t)$ — **[Lokshtanov et al. '10]**

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OUR CONTRIBUTION

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We further improve the state-of-the-art for the subset sum by providing the fastest **deterministic** pseudopolynomial time algorithm for the problem.

Main Theorem [Koiliaris & Xu '17]. *The subset sum problem can be decided in* $\widetilde{O}(\min\{\sqrt{n}t, t^{4/3}\})$ *time.*

Concurrent to our work, Bringmann showed that if **randomization** is allowed the subset sum problem can be decided in $\widetilde{O}(t)$, with one-sided error probability $1/n$.

[Bringmann '17]

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Computing all subset sums for some $u \geq t$ also answers the subset sum problem with target value t.

ALGORITHM

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- Combine them together using FFT

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- If sets S, T lie in a short and light interval, then one can combine their subset sums quickly as their total sum will be small.
- If sets S, T lie in a long and heavy interval, then one can combine their subset sums quickly by ignoring most of the sums as they exceed the upper bound.

NOTATION

- $[\![x : y]\!] = \{x, x+1, \ldots, y\}$ is the set of integers in the interval $[x, y]$.

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$$\sum(S) = \left\{ \sum_{t \in T} t \,\middle|\, T \subseteq S \right\}.$$

- If P and Q form a partition of a set S, then $\sum(P) \oplus \sum(Q) = \sum(S)$.

Lemma [Short and Light]. *Given a set of positive integers S with total sum σ, one can compute the set of all subset sums $\sum(S) \cap [\![0 : u]\!]$ in $O(\sigma \log \sigma \log n)$ time.*

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- Since the divide-and-conquer recursion has $\log n$ levels, we get the $O(\log n(\sigma \log \sigma))$ promised running time.

$\square$

Lemma [Long and Heavy]. *Given a set* $S \subseteq [\![x : x+\ell]\!]$ *of size* n*, computing the set* $\sum(S) \cap [\![0 : u]\!]$ *takes* $\widetilde{O}\big((u/x)^2\ell\big)$ *time.*

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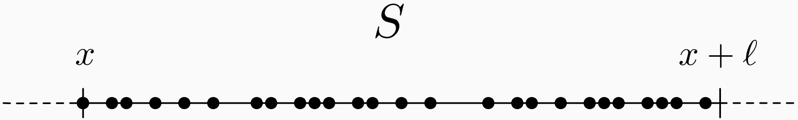
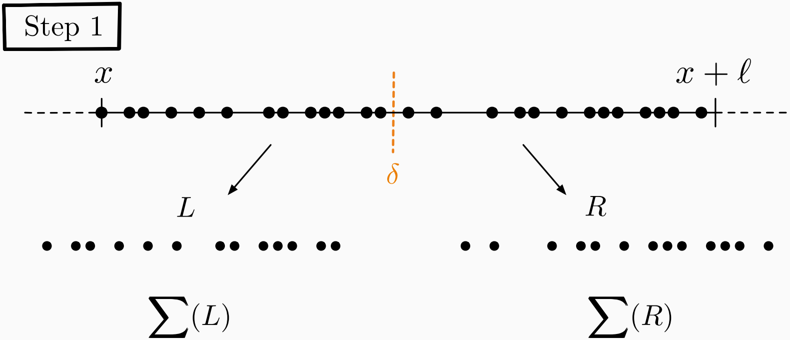

S
x
$x + \ell$


Step 1
x
$x + \ell$
δ
L
R
$\sum(L)$
$\sum(R)$

$\boxed{\text{Step 2}}$

$$\sum(L), \sum(R)$$

$$\begin{array}{c}\Cup \\ z = ix + j\end{array}$$

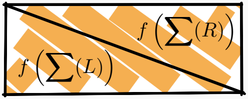
$\boxed{\text{Step 2}}$

$$\sum(L), \sum(R)$$

$$\underset{z}{\Cup}\! = ix + j \xrightarrow{f} (i, j) \in [\![0 : k]\!] \times [\![0 : \ell k]\!]$$

$\boxed{\text{Step 2}}$

$\sum(L), \sum(R) \xrightarrow{f}$



$\Cup$

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Higher # dimensions but shorter intervals

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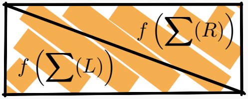
$\sum(L), \sum(R) \xrightarrow{f}$ 

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Higher # dimensions but shorter intervals

$$\text{FFT} \Downarrow \text{ in } \widetilde{O}(\ell k^2)$$

$$f\left(\sum(L)\right) \oplus f\left(\sum(R)\right)$$

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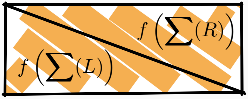

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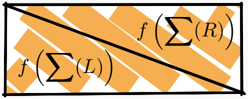
$$\sum(L \cup R) \xleftarrow{f^{-1}} f\left(\sum(L)\right) \oplus f\left(\sum(R)\right)$$

$\boxed{\text{Step 2}}$

$$\textstyle\sum(L), \sum(R) \xrightarrow{f}$$



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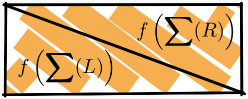

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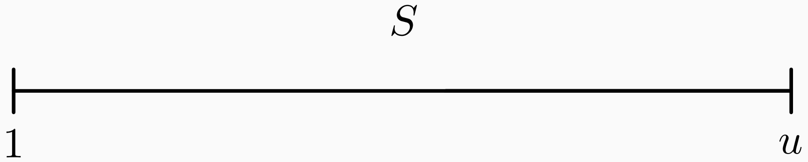
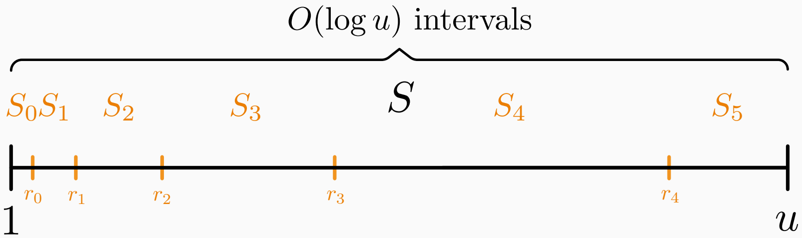
and if $k = \left\lfloor \frac{u}{x} \right\rfloor \implies \widetilde{O}\left(\left(\frac{u}{x}\right)^2 \ell\right)$.

THE ALGORITHM

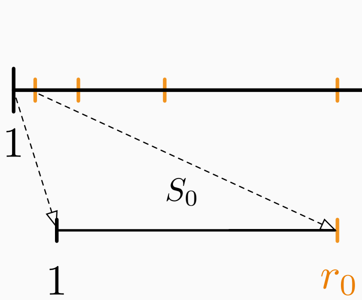


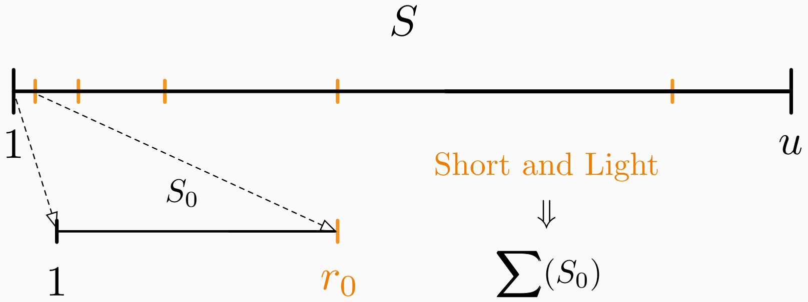


$$S_0 = S \cap [\![1 : r_0]\!]$$
$$S_i = S \cap [\![r_{i-1} + 1 : r_i]\!]$$
$$|S_i| = n_i$$

$r_0 \geq 1$

$r_i = \lfloor 2^i r_0 \rfloor$


S
1
u
S_0
1
r_0



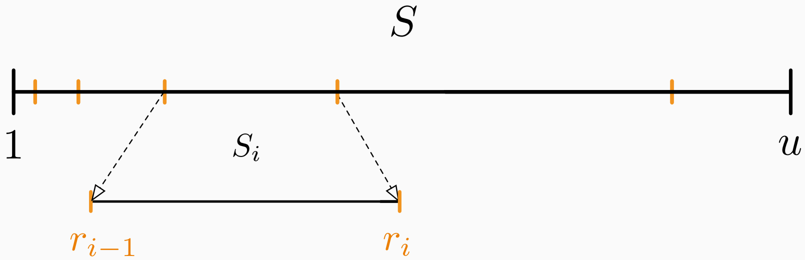
Short and Light

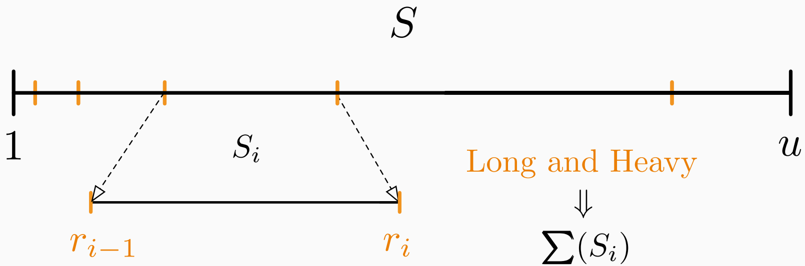
$$\Downarrow$$

$$\sum(S_0)$$

in

$$\widetilde{O}(n_0 r_0) = \widetilde{O}\left(\min\{n, r_0\} r_0\right)$$


S
1
u
S_i
r_{i-1}
r_i



Long and Heavy

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$$\widetilde{O}\left(\left(\frac{u}{r_{i-1}}\right)^2 \ell_i\right) = \widetilde{O}\left(\frac{u^2}{r_{i-1}}\right)$$

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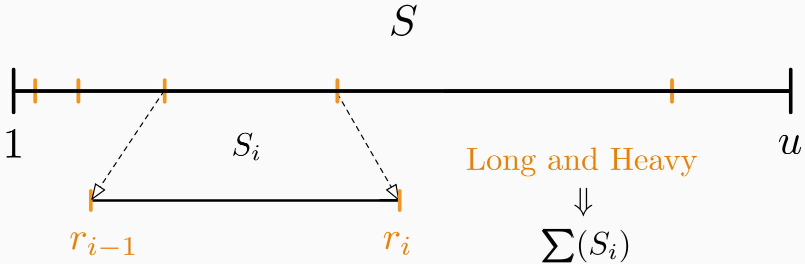

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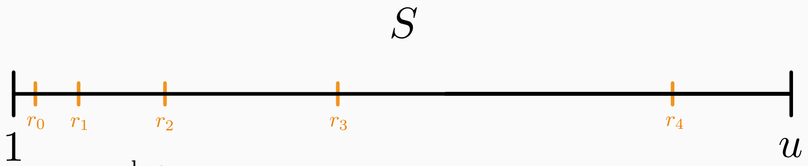
$$\widetilde{O}\left(\left(\frac{u}{r_{i-1}}\right)^2 \ell_i\right) = \widetilde{O}\left(\frac{u^2}{r_{i-1}}\right)$$

$$\implies \widetilde{O}\left(\frac{u^2}{r_0}\right) \text{ for all intervals}$$



$$\bigoplus_{i=0}^{\log u} \sum(S_i) \cap [\![1 : u]\!] = \sum(S) \cap [\![1 : u]\!]$$

$$\Downarrow$$

$$O\left(\log u(u \log u)\right) = \widetilde{O}(u)$$

$$S \rightsquigarrow \sum(S) \cap [\![1 : u]\!]$$

In total time:

$$\widetilde{O}\left(\min\{n, r_0\} r_0 + \frac{u^2}{r_0} + u\right)$$

$r_0 = \dfrac{u}{\sqrt{n}}$

$r_0 = u^{2/3}$

$\widetilde{O}\left(\sqrt{n}u\right)$

$\widetilde{O}\left(u^{4/3}\right)$

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Not trivial!

The challenge is that the previous algorithm throws away many sums that fall outside of $[\![1 : u]\!]$ during its execution, but this can no longer be done for finite cyclic groups, since these sums stay in the group and as such must be accounted for.

FUTURE WORK

Can we use these observations on other similar number-based problems?

OPEN PROBLEMS

Can we use these observations on other similar number-based problems?

Is there a deterministic $\widetilde{O}(t)$ time algorithm for the subset sum problem matching its conditional lower bound?

Thank you